§10 Completions

§10.1 Topologies and completions.

G = topological abelian group

= top. space G + cont. gp str. $G \times G \xrightarrow{\text{mult}} G$ & $G \xrightarrow{\text{inv}} G$

$x, y \mapsto x+y$ $\qquad$ $x \mapsto -x$

(a topological space X is called <u>Hausdorff</u>, if $\forall$ $x_1, x_2 \in X$ $\exists$ U_i s.t. $U_i \cap U_j = \phi$.)

Fact 1) G = Hausdorff $\Leftrightarrow$ $\{0\}$ is closed in G,

$\Leftrightarrow$ diagonal closed $\Leftrightarrow$

2) the topology of G is uniquely determined by the neighborhoods of 0 in G.

$$H_g := \bigcap_{U \ni g : \text{open}} U \qquad H := H_0$$

Fact 1) $H_g = g + H$

2) $\forall h \in H \Rightarrow H = h + H$

3) $H = -H$

Pf: 1). $H_g \overset{\text{所有开集是平移得来的.}}{=} \bigcap_{V \ni 0 : \text{open}} (g + V) = g + \bigcap_{V \ni 0 : \text{open}} V = g + H$

2). $H = \bigcap_{V \ni 0 : \text{open}} V \supseteq \left(\bigcap_{V \ni 0 : \text{open}} V \right) \cap \left(\bigcap_{\substack{V \ni 0 \\ V \ni h}:\text{open}} V \right)$

$= \bigcap_{V \ni h : \text{open}} V = H_h = h + H$

3). $H = \bigcap_{V \ni 0 : \text{open}} (-V) = - \bigcap_{V \ni 0 : \text{open}} V = -H$

Lemma 6.0.1. $H := \bigcap_{U \ni 0 \text{ open}} U$

i) $H < G$

ii) $H = \overline{\{0\}}$

iii) $G/H =$ Hausdorff.

iv) $G =$ Hausdorff $\Leftrightarrow H = 0$

Pf i) $h_1 - h_2 \overset{(3)}{\in} h_1 + H \overset{(2)}{=} H$

ii) $x \in \overline{\{0\}} \Leftrightarrow \left(0 \in U^c \overset{U:\text{open}}{\Rightarrow} x \in U^c \right)$

$\Leftrightarrow \left(x \in U \overset{U:\text{open}}{\Rightarrow} 0 \in U \right)$

$$\Leftrightarrow \quad 0 \in H_x \overset{(1)}{=} x + H$$

$$\Leftrightarrow \quad x \in -H \overset{(3)}{=} H$$

iii) G/H = Hausdorff $\Leftrightarrow$ $\{H\} \in G/H$ is closed

$\Leftrightarrow$ $H \subseteq G$ is closed

iv) G = Hausdorff $\Leftrightarrow$ $\{0\}$ = closed

$$\Leftrightarrow \quad \{0\} = \overline{\{0\}} = H$$

$$\Leftrightarrow \quad H = 0. \qquad \square$$

Cauchy sequence in G

$$C = \{ (x_1, x_2, \cdots) \mid x_i \in G,\ \forall U \ni 0 \text{ open}.\ \exists N \text{ s.t. } x_i - x_j \in U \ \forall i,j > N \}$$

$$(x_\nu) \sim (y_\nu) \overset{\text{def}}{\Longleftrightarrow} x_\nu - y_\nu \to 0 \text{ in } G$$

i.e. $\forall U \ni 0.\ \exists N$ s.t. $x_\nu - y_\nu \in U \ \forall \nu > N$

Completion of G

$$\hat{G} := C/\sim \qquad [(x_\nu)] + [(y_\nu)] := [(x_\nu + y_\nu)]$$

$\phi : G \to \hat{G} \qquad g \mapsto [(g, g, \cdots)]$

e.g. $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Fact: 1) $\ker \phi = \bigcap_{U \ni 0 \text{ open}} U$

2) $\phi = \text{inj} \Leftrightarrow G = \text{Hausdorff}$.

pf: $x \in \ker \phi \Leftrightarrow (x, x, \cdots) \sim (0, 0, \cdots)$

$\Leftrightarrow x \to 0$ in G

$\Leftrightarrow x \in U, \ \forall U \ni 0$ open

- $\forall f : G \to H$ continuous $\Rightarrow \hat{f} : \hat{G} \to \hat{H}$
- $G \xrightarrow{f} H \xrightarrow{g} K \Rightarrow \hat{G} \xrightarrow{\hat{f}} \hat{H} \xrightarrow{\hat{g}} \hat{K}$

Let G be a top. gp with system of neighborhoods consisting of subgps

$$G = G_0 > G_1 > G_2 > \cdots.$$

e.g. p-adic topology on $\mathbb{Z}$:

$$\mathbb{Z} \supseteq p\mathbb{Z} \supseteq p^2\mathbb{Z} \supseteq \cdots$$

Fact: G_n are both open and closed $\forall n$.

Pf: open ✓

closed: $G_n = \text{open} \Rightarrow g + G_n = \text{open} \ \forall g \in G_n$

$$\Rightarrow \bigcup_{g \notin G_n} (g + G_n) = \text{open}$$

$$\Rightarrow G_n = G \Big\backslash \bigcup_{g \notin G_n} (g + G_n) \quad \text{closed.}$$

<u>Inverse limits</u>

Inverse system

$$\cdots A_{n+1} \xrightarrow{\theta_{n+1}} A_n \xrightarrow{\theta_n} A_{n-1} \rightarrow \cdots \xrightarrow{\theta_2} A_1 \xrightarrow{\theta_1} A_0$$

$$\varprojlim_n A_n := \left\{ (a_n) \in \prod_{n=0}^{\infty} A_n \;\middle|\; \theta_{n+1}(a_{n+1}) = a_n \right\}$$

$$(a_n) + (b_n) := (a_n + b_n)$$

<u>Lemma</u> (<u>purely algebraic definition of completion</u>)

$$\hat{G} \cong \varprojlim G/G_n$$

In particular, $\varprojlim G/G_n$ doesn't depend on the choice of $\{G_n\}$.

Pf: $\forall \ [(x_\nu)] \in \hat{G}, \ \forall n \geq 0$

$$\xi_n := x_\nu + G_n \qquad \nu \gg 0.$$

$$\Rightarrow \quad \theta_{n+1}(\xi_{n+1}) = \xi_n \qquad \theta_{n+1} : G/G_{n+1} \to G/G_n$$

$$\Rightarrow \quad (\xi_n)_n \in \varprojlim G/G_n$$

$$[(x_\nu)] \in \ker \Leftrightarrow (\xi_n)_n = 0$$

$$\Leftrightarrow \quad \forall n,\ x_\nu \in G_n \quad \forall\ \nu \gg 0$$

$$\Leftrightarrow \quad (x_1, x_2, \cdots) \sim (0, 0, \cdots)$$

$$\Leftrightarrow \quad [(x_\nu)] = 0.$$

$\forall\ (a_\nu) \in \varprojlim G/G_n$

$$\forall\ x_\nu \in a_\nu \quad \forall \nu \ \Rightarrow \ x_{\nu+1} - x_\nu \in G_\nu$$

$$\Rightarrow \quad x_{\nu+\omega} - x_\nu = (x_{\nu+\omega} - x_{\nu+\omega-1}) + (x_{\nu+\omega-1} - x_{\nu+\omega-2})$$
$$+ \cdots + (x_{\nu+1} - x_\nu) \in G_\nu$$

$$\Rightarrow [(x_\nu)] \in \hat{G} \quad \text{with} \quad [(x_\nu)] \mapsto (a_\nu).$$

$\cdots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0$ is called surjective system

e.g. $\quad \cdots \twoheadrightarrow G/G_2 \twoheadrightarrow G/G_1 \twoheadrightarrow G/G_0.$

exact sequence of inverse systems

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_2 \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & A_0 \arrow[r] & B_0 \arrow[r] & C_0 \arrow[r] & 0 & \text{exact}
\end{tikzcd}
$$

$0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ exact.

$\Rightarrow$ homomorphisms

(*) $\quad 0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$

(not always exact!)

Prop 10.2: 1) $\varprojlim$ is left exact. i.e.

$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$ exact

2) If $\{A_n\}$ is a surjective system, then (*) is exact.

Pf: $A := \prod_{n=0}^{\infty} A_n$, $d^A : A \to A$. $(a_n)_n \mapsto (a_n - \theta_{n+1}(a_{n+1}))_n$

$$\Rightarrow \varprojlim A_n = \ker d^A$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "d^A"] & B \arrow[r] \arrow[d, "d^B"] & C \arrow[r] \arrow[d, "d^C"] & 0 & \text{exact} \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 & \text{exact}
\end{tikzcd}$$

$$\Rightarrow \quad 0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \operatorname{Coker} d^A$$

$A_{n+1} \twoheadrightarrow A_n \ \forall n \Rightarrow d^A =$ surj

$\Rightarrow \operatorname{Coker} d^A \Rightarrow \checkmark$

$\forall (a_n)_n \in A$ $\quad x_0 := 0$ find x_n inductively

$$\theta_{n+1}(x_n) = x_{n-1} - a_{n-1}$$

$$\Rightarrow \quad d^A((x_n)) = (a_n)$$

Cor 10.3. $0 \to G' \to G \xrightarrow{p} G'' \to 0$ exact

topologies $G \rightsquigarrow \{G_n\}$

induced top. $G' \mapsto G' \cap G_n$

$G'' \mapsto pG_n$

Then $0 \to \widehat{G'} \to \widehat{G} \to \widehat{G''} \to 0$ exact.

Pf: $0 \to \dfrac{G}{G' \cap G_n} \to \dfrac{G}{G_n} \to \dfrac{G''}{pG_n} \to 0$ exact $\square$

Cor 10.4 i) $\widehat{G_n}$ is a subgroup of $\widehat{G}$

ii) $\widehat{G}/\widehat{G_n} \simeq G/G_n$

Pf: $0 \to G_n \to G \to G/G_n \to 0$ exact

$\Rightarrow 0 \to \widehat{G_n} \to \widehat{G} \to G/G_n \to 0$ exact $\square$

Prop 10.5 $\hat{\hat{G}} \cong \widehat{G}$

Pf: $\hat{\hat{G}} = \varprojlim \widehat{G}/\widehat{G_n} = \varprojlim G/G_n = \widehat{G}$ $\square$

Def: G is called complete if $G \xrightarrow{\sim} \hat{G}$.

- completion of G is complete.
- complete $\Rightarrow$ hausdorff.

__$\mathfrak{a}$-adic topology__:

- A = ring, $\mathfrak{a} \triangleleft A$ $\quad G = A$, $G_n := \mathfrak{a}^n$

$\rightsquigarrow$ $\mathfrak{a}$-adic topology on A (defined by G_n)

$\rightsquigarrow$ A is a topological ring (i.e. ring operators are cont.)

$\rightsquigarrow$ completion $\hat{A}$ = topological ring.

$\rightsquigarrow$ $\phi : A \to \hat{A}$ $\qquad \ker \phi = \bigcap \mathfrak{a}^n$

- M = A-module. $\quad G = M$, $G_n = \mathfrak{a}^n M$

$\rightsquigarrow$ $\mathfrak{a}$-topology on M. (defined by G_n)

$\rightsquigarrow$ completion $\hat{M}$ = topological $\hat{A}$-module

(i.e. $\hat{A} \times \hat{M} \to \hat{M}$ cont.)

- $\forall\ f : M \to N$

$$f(\pi^n M) = \pi^n f(M) \subseteq \pi^n N$$

$\Rightarrow f$ is continuous

$\Rightarrow \hat{f} : \hat{M} \to \hat{N}$

§10.2 Filtrations

$M = A$-module.

- filtration of M

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \quad (M_i \text{ submodules})$$

denote by (M_n)

- $\underline{\mathfrak{a}\text{-filtration}}$, if $\mathfrak{a} M_n \subseteq M_{n+1} \quad \forall n$
- $\underline{\text{stable } \mathfrak{a}\text{-filtration}}$, if it is $\mathfrak{a}$-filtration, and

$$\mathfrak{a} M_n = M_{n+1} \qquad n \gg 0.$$

e.g. $(\mathfrak{a}^n M)_n$ is a stable $\mathfrak{a}$-filtration.

<u>Lem 10.6</u> any two stable $\mathfrak{a}$-filtrations has bounded difference.

i.e. $M_n \subseteq M \supseteq M_n'$ stable,

$$\Rightarrow \exists\, n_0 \in \mathbb{N} \text{ s.t. } \begin{cases} M_{n+n_0} \subseteq M_n' \\ M_{n+n_0}' \subseteq M_n \end{cases}$$

12 In particular, induce the same top. on M. ($\mathfrak{a}$-topology)

Pf: WMA: $M_n' = \mathfrak{a}^n M$.

$\exists\, n_0$ s.t. $\mathfrak{a} M_n = M_{n+1} \quad \forall\, n \geq n_0$

$$M'_{n+n_0} = \mathfrak{a}^{n+n_0} M \subseteq \mathfrak{a}^n M \subseteq M_n$$

$$M_{n+n_0} = \mathfrak{a}^n M_{n_0} \subseteq \mathfrak{a}^n M = M'_n \qquad \square$$

§ 10.3 graded rings and modules

ring subgroup

graded ring A : $A = \bigoplus_{n=0}^{\infty} A_n$ satisfying

$$A_m \cdot A_n \subseteq A_{m+n} \quad \forall\, m,n \geq 0.$$

Fact : 1) $A_0 \subseteq A$ subring

2) $A_+ := \bigoplus_{n=1}^{\infty} A_n$ ideal of A.

A-module subgroups

graded A-module M : $M = \bigoplus_{n=0}^{\infty} M_n$ satisfying

$$A_m M_n \subseteq M_{m+n}$$

Fact : $M_n = A_0$-module

homogeneous element of degree $n \overset{\text{def}}{\Longleftrightarrow} x \in M_n$.

$\forall\, y \in M \Rightarrow y = \sum_{n=0}^{\infty} y_n,\ y_n \in M_n$

$\hookrightarrow$ homogeneous components of y

homomorphism of graded A-module = A-mod. hom.

$f: M \to N$ s.t. $f(M_n) \subseteq N_n$ $\forall n \geq 0$.

Prop 10.7 A = graded ring. TFAE.

i) A = noetherian

ii) A_0 = noetherian & A = f.g. A_0-alg.

pf: ii) $\Rightarrow$ i) clear (Hilbert's basis thm (7.6))

i) $\Rightarrow$ ii) $A_0 \cong A/A_+ \Rightarrow A_0$ = noeth.

$A_+ \triangleleft A \Rightarrow$ f.g. $\Rightarrow \exists\, x_1, \cdots, x_s \in A_+$ s.t.

$$A_+ = \sum_{i=1}^{s} A \cdot x_i \quad (\text{WMA: } x_i \text{ homog.})$$

$A' := A_0[x_1, \cdots, x_s] \subseteq A$.

We show $A_n \subseteq A'$ inductively:

$n=0$ ✓ assume $A_{n-1} \subseteq A'$.

$\forall\, y \in A_n \Rightarrow y = \sum_{i=1}^{s} a_i x_i$ $\deg a_i = n - \deg x_i$

$$\Rightarrow y \in \sum_{i=1}^{s} A' \cdot x_i \subseteq A'$$

$A = $ ring (not graded). $\mathfrak{a} \lhd A$

$\Rightarrow$ graded ring: $A^* = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n$ $\qquad \boxed{A \subseteq A^* \subseteq A[x]}$

$M = A$-module with $\mathfrak{a}$-filtration M_n.

$\Rightarrow$ graded A^*-module: $M^* = \bigoplus_{n=0}^{\infty} M_n$

Fact: $A =$ noeth. $\Rightarrow A^* =$ noeth.

Lem 10.8: $A =$ noeth, $M =$ f.g. A-module. $(M_n) = \mathfrak{a}$-filtration

TFAE:

i) $M^* =$ f.g. A^*-mod.

ii) $(M_n) =$ stable.

Pf: $M_n =$ f.g. A-mod $\Rightarrow Q_n := \bigoplus_{r=0}^{n} M_r =$ f.g. A-mod

$\Rightarrow M_n^* = A^* Q_n = \left(\bigoplus_{r=0}^{n} M_r\right) \oplus \left(\bigoplus_{r=1}^{\infty} \mathfrak{a}^r M_n\right)$

f.g. A^*-mod

- $M_1^* \subseteq M_2^* \subseteq \cdots \subseteq M^*$
- $M^* = \bigcup_{i=1}^{\infty} M_i^*$

$M^* =$ f.g. A^*-mod $\Leftrightarrow \{M_i^*\}$ stop

$\Leftrightarrow \quad M^* = M^*_{n_0}$ for some n_0

$\Leftrightarrow \quad M_{n_0+r} = \mathfrak{a}^r M_{n_0} \quad \forall r \geq 0$

$\Leftrightarrow$ stable

<u>Prop 10.9</u> (Artin-Rees Lemma) A = noeth. $\mathfrak{a} \triangleleft A$, M = f.g. A-mod. (M_n) = stable $\mathfrak{a}$-fil. of M.

$M' \subseteq M$ submod $\Rightarrow$ $(M' \cap M_n)$ = stable $\mathfrak{a}$-fil. of M

pf: $\mathfrak{a}(M' \cap M_n) \subseteq \mathfrak{a}M' \cap \mathfrak{a}M_n \subseteq M' \cap M_{n+1} \Rightarrow \mathfrak{a}$-fil.

(M_n) = stable $\Rightarrow M^*$ = f.g. A^*-mod

$\Rightarrow M'^*$ = f.g. A^*-mod (A^* = noeth.)

$\Rightarrow (M' \cap M_n)$ = stable. $\square$

<u>Cor 10.10</u> (usual version) $\exists k$ s.t. $\mathfrak{a}^n M \cap M' = \mathfrak{a}^{n-k}((\mathfrak{a}^k M) \cap M') \ \forall n \geq k$.

Pf: $M_n := \mathfrak{a}^n M$ $\square$

<u>Thm 10.11</u> (another version): A = noeth. $\mathfrak{a} \triangleleft A$, M = f.g. $M' \subseteq M$ submod.

$\Rightarrow \mathfrak{a}^n M'$ & $\mathfrak{a}^n M \cap M'$ has bounded difference

$\Rightarrow \mathfrak{a}$-top. of M' = induced top by $\mathfrak{a}$-top of M'.

Prop 10.12 (Exactness of adic completions) A = noeth. $\mathfrak{a} \triangleleft A$

M = f.g. A-mod. $0 \to M' \to M \to M'' \to 0$ exact. Then

$$0 \to \underbrace{\widehat{M'} \to \widehat{M} \to \widehat{M''}}_{\mathfrak{a}\text{-adic completions}} \to 0 \text{ exact}$$

Pf : (10.11) + (10.3) $\square$

$M \rightsquigarrow$ two $\hat{A}$- module

- $M \otimes_A \hat{A}$ & $\hat{M}$ (relation ?)

natural $\hat{A}$-hom :

$$M \to \hat{M} \quad m \mapsto (m, m, \cdots)$$

and

$$\hat{A} \otimes_A M \to \hat{A} \otimes_A \hat{M} \to \hat{A} \otimes_{\hat{A}} \hat{M} \xrightarrow{\sim} \hat{M}$$

$$(a_1, a_2, \cdots) \otimes_A m \longmapsto (a_1 m, a_2 m, \cdots)$$

Prop 10.13 : A = ring. M = f.g.

i) $\hat{A} \otimes_A M \twoheadrightarrow \hat{M}$

ii) A = noeth. $\Rightarrow$ $\hat{A} \otimes_A M \xrightarrow{\sim} \hat{M}$

Pf: $0 \to N \rightarrowtail \underset{\displaystyle F}{\overset{\|}{A^{\oplus n}}} \twoheadrightarrow M \to 0$ exact

$$\begin{array}{ccccccc} & \hat{A}\otimes N & \to & \hat{A}\otimes_A F & \twoheadrightarrow & \hat{A}\otimes_A M & \to 0 \\ & \downarrow \varphi_N & & \downarrow \cong & & \downarrow \varphi_M & \\ \Rightarrow \quad 0 \to & \hat{N} & \to & \hat{F} & \xrightarrow{\pi} & \hat{M} & \to 0 \end{array} \qquad (10.3)$$

$\pi = $ surj $\Rightarrow \varphi_M = $ surj. $\Rightarrow \varphi_N = $ surj. $\Rightarrow \varphi_M = $ inj. $\square$

Prop 10.14. $A = $ noeth. $\mathfrak{a} \triangleleft A \Rightarrow \hat{A} = $ flat A-alg.

Pf. (10.12) + (10.13) + chapter 2 $\square$

Rmk: • $M \mapsto \hat{M}$ is NOT exact for non-f.g. modules!

• two functor coincide on f.g. modules.

Prop 10.15 (elementary properties of $\hat{A}$). $A = $ noeth. $\mathfrak{a} \triangleleft A$.

i) $\hat{\mathfrak{a}} = \hat{A}\mathfrak{a} \cong \hat{A} \otimes_A \mathfrak{a}$

ii). $\widehat{\mathfrak{a}^n} = (\hat{\mathfrak{a}})^n$

iii). $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$

iv). $\hat{\mathfrak{a}} \subseteq \mathrm{Rad}(\hat{A})$ (Jacobson radical) pf: i). A = noeth. $\Rightarrow$ $\mathfrak{a}$ = f.g. $\overset{(10.13)}{\Rightarrow}$ $\hat{A} \otimes_A \mathfrak{a} \xrightarrow{\sim} \hat{\mathfrak{a}}$, $\hat{A}\mathfrak{a}$

ii). $\widehat{\mathfrak{a}^n} \overset{(i)}{=} \hat{A}\mathfrak{a}^n \overset{1.18}{=} (\hat{A}\mathfrak{a})^n \overset{(i)}{=} (\hat{\mathfrak{a}})^n$

iii). ii) $\overset{(10.4)}{\Rightarrow}$

$$\begin{tikzcd} \hat{A}/\hat{\mathfrak{a}}^n \arrow[r, phantom, "\cong"] & A/\mathfrak{a}^n \\ \hat{A}/\hat{\mathfrak{a}}^{n+1} \arrow[u, "\hat{\varphi}_{n+1}"] \arrow[r, "\cong"] & A/\mathfrak{a}^{n+1} \arrow[u, "\varphi_{n+1}"] \end{tikzcd}$$

$\Rightarrow$ $\ker \hat{\varphi}_{n+1} \cong \ker \varphi_n$, $\ker \hat{\varphi}_{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$, $\ker \varphi_n \cong \mathfrak{a}^n/\mathfrak{a}^{n+1}$

iv). $\forall x \in \hat{\mathfrak{a}}$ $\Rightarrow$ $(1-x)^{-1} = 1 + x + x^2 + \cdots \in \hat{A}$ $\quad \forall x \in \hat{\mathfrak{a}}$

$\overset{(1.9)}{\Rightarrow}$ $\hat{\mathfrak{a}} \subseteq \mathrm{Rad}(\hat{A})$

Prop 10.16 (A, m) = noeth. + local. $\hat{A}$ = m-adic completion.

$\Rightarrow$ $(\hat{A}, \hat{m})$ = local

pf: $\hat{A}/\hat{m} \cong A/m \Rightarrow \hat{m}$ = maximal

$\hat{m} \subseteq \mathrm{Rad}(\hat{A}) \Rightarrow \hat{m}$ = unique maximal ideal. $\square$

取完备化丢失多少？

Thm 10.17 (Krull's thm) A = noeth. $\mathfrak{a}$ = ideal, M = f.g.

$$\ker(M \to \hat{M}) = \{x \in M \mid (1+\alpha)x = 0 \text{ for some } \alpha \in \mathfrak{a}\}$$

Pf: $E := \ker(M \to \hat{M}) = \bigcap_{n=1}^{\infty} \mathfrak{a}^n M$

RHS $\subseteq$ LHS: $\forall\ m \in \text{RHS} \Rightarrow m = -\alpha m$ for some $\alpha \in \mathfrak{a}$

$\Rightarrow m = (-\alpha)^n \cdot m \in \mathfrak{a}^n M \ \forall n$

$\Rightarrow m \in \text{LHS}$

LHS $\subseteq$ RHS: restr. top on E = trivial $\Rightarrow \mathfrak{a}E = E$

$E \subseteq M \Rightarrow E = $ f.g.

$\overset{2.5}{\Rightarrow} \exists\, \alpha \in \mathfrak{a}$ s.t. $(1-\alpha)E = 0$. $\square$

Rmk: 1) $S := 1 + \mathfrak{a}$

$(10.17) \Rightarrow \ker(A \to \hat{A}) = \ker(A \to S^{-1}A)$

$\Rightarrow S^{-1}A \hookrightarrow \hat{A}$ inj.

2) Krull's thm false, if $A \neq$ noeth.

A = ring of $\mathcal{C}^\infty$ functions on $\mathbb{R}$.

$\mathfrak{a} = \{ f \in A \mid f(0) = 0 \} \trianglelefteq A$

$\ker(A \to \hat{A}) = \bigcap_{n=1}^{\infty} \mathfrak{a}^n = \{ f \in A \mid 0 = f(0) = f'(0) = \cdots \}$

$f \in \ker(A \to S^{-1}A) \Leftrightarrow (1+\alpha) f = 0$

$\Leftrightarrow f = 0$ in some eighborhood of 0

$e^{-\frac{1}{x^2}} \in \ker(A - \hat{A}) \setminus \ker(A \to S^{-1}A)$

$\Rightarrow$ Krull's fails for A.

<u>Cor</u> 10.18 A = noeth. domain. $\mathfrak{a} \overset{\neq (1)}{\triangleleft} A$. Then

$$\bigcap \mathfrak{a}^n = 0.$$

Pf: $1+\mathfrak{a}$ contains no zero-divisors $\square$

<u>Cor</u> 10.19. A = noeth. $\mathfrak{a} \subseteq \mathrm{Rad}(A)$. M = f.g.. Then

$$\bigcap \mathfrak{a}^n M = 0$$

(the $\mathfrak{a}$-topology of M is Hausdorff.)

Pf: (1.9) $\Rightarrow$ $1+\alpha \in A^{\times}$ $\square$

<u>Cor</u> 10.20. (A, m) = noeth. local. M = f.g. Then

m-topology of M is Hausdorff.

 In particular, m-top. of A is Hausdorff.

Fact : $\bigcap_{\mathfrak{q}=\mathfrak{m}\text{-primary}} \mathfrak{q} = \bigcap_{\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}} \mathfrak{q} = \bigcap \mathfrak{m}^n = \ker(A \to \hat{A}) = 0$

Cor 10.21. A = noeth. $\mathfrak{p} \trianglelefteq A$ = prime. Then

$$\ker(A \to A_{\mathfrak{p}}) = \bigcap_{\mathfrak{q}=\mathfrak{p}\text{-primary}} \mathfrak{q}$$

Pf: $\bigcap_{\mathfrak{q}'=\mathfrak{p}A_{\mathfrak{p}}\text{-primary}} \mathfrak{q}' = \ker(A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}) = 0$

$$\Rightarrow \ker(A \xrightarrow{\pi} A_{\mathfrak{p}}) = \pi^{-1}(0)$$

$$= \pi^{-1}\Big(\bigcap_{\mathfrak{q}'=\mathfrak{p}A_{\mathfrak{p}}\text{-primary}} \mathfrak{q}'\Big)$$

$$= \bigcap_{\mathfrak{q}=\mathfrak{p}\text{-primary}} \mathfrak{q} \qquad \square$$

§10.4 the associated graded ring.

A = ring, $\mathfrak{a} \trianglelefteq A$. $\Rightarrow$ graded ring

$$G(A) = G_{\mathfrak{a}}(A) := \bigoplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1} \qquad (\mathfrak{a}^0 := A)$$

$$\overline{a_{n_1}} \cdot \overline{a_{n_2}} := \overline{a_{n_1} a_{n_2}} \quad \text{(well-defined)}$$

M = A-module, (M_n) = $\mathfrak{a}$-filtration $\Rightarrow$ graded $G(A)$-module :

$$G(M) = \bigoplus_{n=0}^{\infty} M_n / M_{n+1}$$

$\mathfrak{a}$-filtration

$$\overline{a_{n_1}} \cdot \overline{m_{n_2}} := \overline{a_{n_1} m_{n_2}} \quad \text{(well-defined)}$$

Prop 10.22 A = noeth. $\mathfrak{a} \trianglelefteq A$. Then

i) $G_{\mathfrak{a}}(A)$ = noeth.

ii) $G_{\mathfrak{a}}(A) \cong G_{\hat{\mathfrak{a}}}(\hat{A})$ (as graded rings)

iii) M = f.g. (M_n) = $\mathfrak{a}$-filtration.

(M_n) = stable $\underset{\not\Leftarrow}{\Rightarrow}$ $G(M)$ = f.g. $G(A)$-mod.

Pf: i). $\mathfrak{a} = \sum_{i=1}^{s} A x_i \Rightarrow \mathfrak{a}/\mathfrak{a}^2 = \sum_{i=1}^{s} (A/\mathfrak{a}) \cdot \bar{x}_i$

$\mathfrak{a}^n = \sum_{|I|=n} A x^I \Rightarrow \mathfrak{a}^n/\mathfrak{a}^{n+1} = \sum_{i=1}^{s} (A/\mathfrak{a}) \cdot \bar{x}^I$

$\Rightarrow G(A) = (A/\mathfrak{a})[\bar{x}_1, \cdots, \bar{x}_s]$

$\Rightarrow G(A) = \text{noeth.}$

ii). $(10.15\ \text{iii}) \Rightarrow \mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1} \Rightarrow \checkmark$

iii). stable $\Rightarrow \exists\, n_0$ s.t. $M_{n_0+r} = \mathfrak{a}^r M_{n_0} \quad \forall r \geq 0.$

$\Rightarrow G_{n_0+r}(M) = G_r(A)\, G_{n_0}(M) \quad \forall r \geq 0$

$\Rightarrow G(M)$ generated by $\bigoplus_{n \leq n_0} G_n(M)$

- $M = \text{f.g.} \Rightarrow M = \text{noeth. } A\text{-mod}$

$\Rightarrow G_n(M) = \text{noeth. } A/\mathfrak{a}\text{-mod.}$

$\Rightarrow G_n(M) = \text{f.g. } A/\mathfrak{a}\text{-mod.}$

$\Rightarrow G(M) = \text{f.g. } G(A)\text{-mod.}$

标记: $A = \text{noeth} \Rightarrow \hat{A} = \text{noeth.}$

Lem 10.23: $\phi : A \to B$ hom. of filtered gps (i.e. $\phi(A_n) \subseteq B_n$). Then

i) $G(\phi) = \text{inj.} \Rightarrow \hat{\phi} = \text{inj.}$ $\qquad G(\phi) : G(A) \to G(B)$

ii) $G(\phi) = \text{surj.} \Rightarrow \hat{\phi} = \text{surj}$ $\qquad \hat{\phi} : \hat{A} \to \hat{B}$

pf:

$$\begin{tikzcd}
0 \arrow[r] & A_n/A_{n+1} \arrow[r] \arrow[d, "G_n(\phi)"] & A/A_{n+1} \arrow[r] \arrow[d, "\phi_{n+1}"] & A/A_n \arrow[r] \arrow[d, "\phi_n"] & 0 \\
0 \arrow[r] & B_n/B_{n+1} \arrow[r] & B/B_{n+1} \arrow[r] & B/B_n \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad 0 \to \ker G_n(\phi) \to \ker \phi_{n+1} \to \ker \phi_n \to \operatorname{coker} G_n(\phi) \to \operatorname{coker} \phi_{n+1} \to \operatorname{coker} \phi_n \to 0$$

$$G(\phi) = \text{inj} \overset{\text{inductively}}{\Rightarrow} \phi_n = \text{inj} \Rightarrow \hat{\phi} = \varprojlim \phi_n = \text{inj}$$

$$G(\phi) = \text{surj} \Rightarrow \begin{cases} \phi_n = \text{surj} \\ \ker \phi_{n+1} \twoheadrightarrow \ker \phi_n \Rightarrow (\ker \phi_n) = \text{surj. syst.} \end{cases}$$

$$0 \to \{\ker \phi_n\} \to \{A/A_n\} \xrightarrow{\phi_n} \{B/B_n\} \to 0$$

$$\Rightarrow \hat{\phi} = \text{surj}$$

Prop 10.24 : A = ring, $\mathfrak{a} \triangleleft A$, M = A-mod. (M_n) = $\mathfrak{a}$-filtration.

Suppose A $\mathfrak{a}$-adic complete & M Hausdorff. Then

$G(M)$ = f.g. $G(A)$-module $\Rightarrow$ M = f.g. A-mod.

Pf: $G(M)$ = f.g. $\Rightarrow$ $\exists$ system of homogeneous generators $\xi_1, \cdots, \xi_r$ of $G(M)$

(assume $\deg \xi_i = n_i$
$\forall\, x_i \in \xi_i \in M_{n_i}/M_{n_i+1}$)

$A(n)$ = $\mathfrak{a}$-filtered A-module defined by

$$\underset{\substack{\parallel \\ A}}{A(n)_0} \supseteq \cdots \supseteq \underset{\substack{\parallel \\ A}}{A(n)_n} \supseteq \underset{\substack{\parallel \\ \mathfrak{a}}}{A(n)_{n+1}} \supseteq \underset{\substack{\parallel \\ \mathfrak{a}^2}}{A(n)_{n+2}} \supseteq \cdots$$

$\Rightarrow F = \bigoplus_{i=1}^{r} \underbrace{A(n_i)}_{A \cdot e_i}$ $\mathfrak{a}$-filtered A-module.

$\Rightarrow \phi : F \to M$ homo. of $\mathfrak{a}$-filtered A-modules
$e_i \mapsto x_i$

$\Rightarrow G(\phi) : G(F) \to G(M)$ hom. of $G(A)$-modules.

Construction $\Rightarrow G(\phi)$ = surj $\Rightarrow \hat{\phi}$ = surj

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\alpha"', "\cong"] & M \arrow[d, "\beta"] \\ \hat{F} \arrow[r, "\hat{\phi}"] & \hat{M} \end{tikzcd}$$

$$\left.\begin{array}{l} F = \text{free} \Rightarrow \alpha = \text{iso.} \\ M = \text{Hausdorff} \Rightarrow \beta = \text{inj} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \beta = \text{iso} \\ \phi = \text{surj} \Rightarrow M = \text{f.g.} \end{array}\right.$$

Cor 10.25: $A = \mathfrak{a}$-adic complete. $M =$ f.g with $\mathfrak{a}$-filtration (M_n) s.t. top. Hausdorff

$G(M) =$ noeth. $G(A)$-mod. $\Rightarrow M =$ noeth. A-mod.

Pf: $\forall\, M' \subseteq M$ submodule. $M'_n := M' \cap M_n \Rightarrow \left\{\begin{array}{l} \mathfrak{a}\text{-filtration} \\ \text{Hausdorff.} \end{array}\right.$

$\Rightarrow G(M') \subseteq G(M)$ f.g.

$\overset{G(M)=\text{noeth.}}{\Rightarrow} G(M') =$ f.g. $G(A)$-module

$\overset{(10.24)}{\Rightarrow} M' =$ f.g.

Thm 10.26: $A =$ noeth. $\mathfrak{a} \triangleleft A \Rightarrow \hat{A} =$ noeth.

Pf. $A =$ noeth $\overset{10.22}{\Rightarrow} G_{\hat{\mathfrak{a}}}(\hat{A}) = G_{\mathfrak{a}}(A) =$ noeth. $\overset{10.25}{\Rightarrow} \hat{A} =$ noeth. □

Cor 10.27: $A =$ noeth. $\Rightarrow A[[x_1, \cdots, x_n]] =$ noeth.

Pf: $\mathfrak{a} = (x_1, \cdots, x_n) \triangleleft A[x_1, \cdots, x_n]$ □